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GEOMETRICAL STRUCTURES OF SOME NON-DISTANCE MODELS FOR ASYMMETRIC MDS

Naohito Chino* and Kenichi Shiraiwa**

Geometrical structures of some non-distance models for asymmetric MDS are examined in error-free data measured at a ratio level and a unified geometrical interpretation of these models is provided. These include CASK (Canonical Analysis of SKew symmetry), DEDICOM, GIPSCAL, and HCM (Hermitian Canonical Model). It is shown that these models except for CASK as well as other possible models for square asymmetric proximity data matrix are expressible in terms of finite-dimensional complex Hilbert space under some general condition, and that differences in form of these models depend only on the bases chosen. It is also shown that the Hilbert space structure has an interesting property which traditional distance model does not. Finally it is shown that the general condition relates to an extension of the famous Young Householder theorem to complex Hilbert space.

1. Introduction

In recent years much attention has been attracted to non-distance models for asymmetric MDS. These include CASK (Canonical Analysis of SKew symmetry) proposed by Gower (1977) and Constantine and Gower (1978), DEDICOM (DEcomposition into DIrectional COMponents) by Harshman (1978) and Harshman, Green, Wind, and Lundy (1982), GIPSCAL (Generalized Inner Product multidimensional SCALing) by Chino (1978, 1980, 1990), HCM (Hermitian Canonical Model) proposed originally by Escoufier and Grorud (1980), and reformulated independently in a different way and amplified somewhat by Chino (1991a, 1991b).

Unfortunately however, the geometrical structures of these models have been only partially understood. For example, Harshman (1978) calls DEDICOM a non-spatial model. Chino (1990) does not show the necessary and sufficient condition for GIPSCAL to be a Euclidean space model. Recently, Kiers and Takane (1992) have proposed a generalized version of GIPSCAL and have shown the necessary and sufficient condition for DEDICOM and GIPSCAL to have an equivalent expression. But, their discussions are limited to pointing out that these two models can have such an expression. On the other hand, Chino (1991b) has pointed out that HCM can have some metric properties such as a finite-dimentional complex (f.d.c.) Hilbert space structure under a general condition, but his discussion still lacks a unified geometrical interpretation of these four models. Gower and Zielman (1992) have discussed the non-Euclidean space properties of the symmetric part as

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* Department of Psychology, Aichi Gakuin University, Nisshin-cho, Aichi-gun, Aichi, 470-01 Japan
** Department of Industrial Science and Technology, Science University of Tokyo, 2641 Yamazaki, Noda, Chiba, 278 Japan

well as the skew-symmetric part of a proximity matrix. But they neither discuss the concrete space structures of each of them nor refer to the structures as a whole.

In this paper we examine the geometrical structures of these models for asymmetric MDS *in error-free data measured at a ratio level* using several concepts which are little known to psychometricians, and provide a unified geometrical interpretation of their holistic structures. As a result we relate these models to each other. Results are summarized into the following five conclusions, which are proved in section 2, where the terminology is fully explained:

(C1) Complex counterparts of DEDICOM, GIPSCAL, and HCM all assign a Hermitian form to the Hermitian data matrix $\boldsymbol{H}$ constructed uniquely from an original proximity matrix $\boldsymbol{S}$ such that

$$\boldsymbol{H}=\boldsymbol{S}_s+i\boldsymbol{S}_{sk}, \tag{1}$$

where $\boldsymbol{S}_s$ and $\boldsymbol{S}_{sk}$ are the symmetric part and the skew-symmetric part of the proximity data matrix, respectively. Incidentally, these models can be generally called the *Hermitian form models.*

(C2) The complex counterparts of DEDICOM, GIPSCAL, HCM, and other possible models for $\boldsymbol{H}$ are expressible in terms of finite-dimensional complex Hilbert space if $\boldsymbol{H}$ is positive (negative) semi-definite. Moreover, in this case, objects can be represented in the space.

(C3) The complex counterparts of DEDICOM, GIPSCAL, HCM, and other possible models of $\boldsymbol{H}$ are expressible in terms of an indefinite metric if $\boldsymbol{H}$ is indefinite.

(C4) The Hermitian form in DEDICOM, GIPSCAL, and HCM is composed of a metric tensor and an exterior product called the 2-form, which correspond to the symmetric part and the skew-symmetric part of $\boldsymbol{S}$, respectively. Differences in form of these models depend upon the basis chosen for these two parts. On the one hand, GIPSCAL employs an orthogonal basis for both, while HCM chooses an orthonormal basis for the former and a symplectic basis for the latter. Thus, GIPSCAL and HCM have a standard Euclidean structure if $\boldsymbol{H}$ is positive (negative) semi-definite. On the other hand, DEDICOM does not choose a specific standard basis. CASK uses a symplectic basis, but does not assume any metric tensor.

(C5) We can also choose, for example, a symplectic basis for DEDICOM and GIPSCAL. In this case, the metric tensor is furnished with a weighted Euclidean space structure for DEDICOM and GIPSCAL. The generalized version of GIPSCAL as well as a DEDICOM version proposed recently by Kiers and Takane (1992) are such cases.

These conclusions are proved in the next section. In the third section we shall also prove that, under the general condition stated in *C2*, the Hermitian form models have an interesting property, which traditional (squared) distance model does not. In the fourth section we shall prove that the general condition leads to an extension of the famous Young-Householder theorem (Young & Householder, 1938) to complex Hilbert space.

2. Proofs of the five conclusions

Let us now prove seriatim the five conclusions stated in the introduction section.

First, we define the *complex counterparts* of DEDICOM and GIPSCAL. These definitions can be made simply by putting the purely imaginary number i to the second right-hand terms of the following equations for DEDICOM and GIPSCAL:

$$\boldsymbol{S}=\{s_{jk}\}=\boldsymbol{YAY}^t=\boldsymbol{YA}_s\boldsymbol{Y}^t+\boldsymbol{YA}_{sk}\boldsymbol{Y}^t, \tag{2}$$

and

$$\boldsymbol{S}=a\boldsymbol{ZZ}^t+b\boldsymbol{ZL}_q\boldsymbol{Z}^t+c\boldsymbol{1}_N\boldsymbol{1}_N^t, \tag{3}$$

where $\boldsymbol{S}$ is a proximity matrix of order $(N\times N)$, $\boldsymbol{A}_s$ and $\boldsymbol{A}_{sk}$ are the symmetric part and the skew-symmetric part of the DEDICOM "core" matrix $\boldsymbol{A}$ of order $(p\times p)$ giving the directional relationships among the basic p types or dimensions, and $\boldsymbol{L}_q$ is the special skew-symmetric matrix of order $(q\times q)$ in the GIPSCAL model. Note that the constant c in Eq. (3) is zero, since we have assumed that the data is measured at a ratio level in this paper. In terms of Eq. (1), Eq. (2) becomes

$$\boldsymbol{H}=\{h_{jk}\}=\boldsymbol{YA}_s\boldsymbol{Y}^t+i\boldsymbol{YA}_{sk}\boldsymbol{Y}^t, \tag{4}$$

and Eq. (3) becomes

$$\boldsymbol{H}=a\boldsymbol{ZZ}^t+i(b\boldsymbol{ZL}_q\boldsymbol{Z}^t). \tag{5}$$

From Eqs. (4) and (5), we get

$$h_{jk}=\boldsymbol{y}_j^t\boldsymbol{A}_s\boldsymbol{y}_k+i\ \boldsymbol{y}_j^t\boldsymbol{A}_{sk}\boldsymbol{y}_k, \tag{6}$$

and

$$h_{jk}=a\boldsymbol{z}_j^t\boldsymbol{z}_k+i(b\boldsymbol{z}_j^t\boldsymbol{L}_q\boldsymbol{z}_k), \tag{7}$$

respectively.

As for HCM, there exists an orthonormal eigenbasis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_N\}$ in C^N, which is composed of the eigenvectors of $\boldsymbol{H}$, since the starting matrix $\boldsymbol{H}$ of HCM is Hermitian (Lancaster & Tismenetsky, 1985). Thus we have

$$\boldsymbol{Hu}_i=\lambda_i\boldsymbol{u}_i,\quad 1\leq i\leq N. \tag{8}$$

Here, we have without loss of generality, $\lambda_i\neq 0\ (1\leq i\leq n)$, and $\lambda_{n+1}=\lambda_{n+2}=\cdots=\lambda_N=0$. If we let

$$\boldsymbol{U}=\{\underbrace{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n,}_{n}\quad \underbrace{\boldsymbol{u}_{n+1}, \cdots, \boldsymbol{u}_N}_{N-n}\}=(\boldsymbol{U}_1,\ \boldsymbol{U}_2),$$

then Eq. (8) can also be written as

$$\boldsymbol{HU}=(\boldsymbol{U}_1,\ \boldsymbol{U}_2)\begin{pmatrix}\boldsymbol{\Lambda}, & 0\\ \boldsymbol{0}, & \boldsymbol{0}\end{pmatrix}=\boldsymbol{U}\tilde{\boldsymbol{\Lambda}},$$

where $\Lambda=diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Then we have

$$\boldsymbol{H}=\boldsymbol{U}_1\boldsymbol{\Lambda}\boldsymbol{U}_1^*, \tag{9}$$

since $\boldsymbol{U}$ is unitary. Note that Eq. (9) holds not only for HCM but also for DEDICOM as well as GIPSCAL, since the left-hand sides of Eqs. (4) and (5) are both Hermitian.

Moreover, if we put

$$\boldsymbol{U}_1 = \boldsymbol{U}_r + i\boldsymbol{U}_c, \quad \boldsymbol{X} = (\boldsymbol{U}_r, \ \boldsymbol{U}_c), \tag{10}$$

then we get

$$\boldsymbol{H} = \boldsymbol{X}\boldsymbol{\Omega}_s\boldsymbol{X}^t + i\ \boldsymbol{X}\boldsymbol{\Omega}_{sk}\boldsymbol{X}^t, \tag{11}$$

where

$$\boldsymbol{\Omega}_s = \begin{pmatrix} \boldsymbol{\Lambda}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{\Lambda} \end{pmatrix}, \quad \boldsymbol{\Omega}_{sk} = \begin{pmatrix} \boldsymbol{0}, & -\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}, & \boldsymbol{0} \end{pmatrix}, \tag{12}$$

and $\boldsymbol{U}_r$ and $\boldsymbol{U}_c$ are the real part and the imaginary part of $\boldsymbol{U}_1$, respectively.

From Eq. (11) we have

$$h_{jk} = \boldsymbol{x}_j^t\boldsymbol{\Omega}_s\boldsymbol{x}_k + i\ \boldsymbol{x}_j^t\boldsymbol{\Omega}_{sk}\boldsymbol{x}_k. \tag{13}$$

Then, conclusion $C1$ is evident. For, putting $\varphi(\boldsymbol{\zeta}, \boldsymbol{\tau}) = \boldsymbol{\zeta}\boldsymbol{\Lambda}\boldsymbol{\tau}^*$, φ satisfies the following properties of the Hermitian form (Cristescu, 1977 ; Lancaster and Tismenetsky, 1985) :

$$\text{(i)} \quad \varphi(\boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2, \boldsymbol{\tau}) = \varphi(\boldsymbol{\zeta}_1, \boldsymbol{\tau}) + \varphi(\boldsymbol{\zeta}_2, \boldsymbol{\tau}),$$
$$\text{(ii)} \quad \varphi(a\boldsymbol{\zeta}, \boldsymbol{\tau}) = a\varphi(\boldsymbol{\zeta}, \boldsymbol{\tau}),$$
$$\text{(iii)} \quad \varphi(\boldsymbol{\zeta}, \boldsymbol{\tau}) = \overline{\varphi(\boldsymbol{\tau}, \boldsymbol{\zeta})},$$

where $\boldsymbol{\tau}$, $\boldsymbol{\zeta}$ are row vectors in an n-dimensional complex vector space, and $\overline{\varphi(\boldsymbol{\tau}, \boldsymbol{\zeta})}$ is the complex conjugate of $\varphi(\boldsymbol{\tau}, \boldsymbol{\zeta})$. Moreover, from Eq. (9) we get $h_{jk} = \varphi(\boldsymbol{v}_j, \boldsymbol{v}_k)$, where $\boldsymbol{v}_j$, $\boldsymbol{v}_k$ are the row vectors corresponding to the j-th row and the k-th row of $\boldsymbol{U}_1$. Here, it should be emphasized that Eq. (9) holds not only for DEDICOM, GIPSCAL, and HCM but also for other possible models for $\boldsymbol{H}$.

In general, a Hermitian form is said to be positive-definite if $\varphi(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq 0$ for any $\boldsymbol{\zeta}$, and is called a *(Hermitian) scalar product* if $\varphi(\boldsymbol{\zeta}, \boldsymbol{\zeta}) > 0$ for any $\boldsymbol{\zeta} \neq \boldsymbol{0}$. This means that we can define a *seminorm* on a vector space if, for example, $\varphi(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq 0$ for any $\boldsymbol{\zeta}$ in such a way that

$$\| \boldsymbol{\zeta} \| = \sqrt{\varphi(\boldsymbol{\zeta}, \boldsymbol{\zeta})}. \tag{14}$$

In particular, if φ is positive for any $\boldsymbol{\zeta} \neq \boldsymbol{0}$, $\| \boldsymbol{\zeta} \|$ defines a *norm*. Since a normed vector space is a *pre-Hilbert space* if and only if the norm is generated by a scalar product, DEDICOM, GIPSCAL, HCM, and other possible models for $\boldsymbol{H}$ are expressible in terms of pre-Hilbert space. Furthermore, these models can be stated in terms of *f.d.c. Hilbert space*, since a finite-dimensional complex space is always *complete*. Of course, the positive semi-difiniteness of $\boldsymbol{H}$ is a necessary and sufficient condition for these models to be expressible in terms of complex Hilbert space. The proof of conclusion $C2$ is straightforward for HCM. It should be understood that the sign of $\boldsymbol{H}$ is altered when $\boldsymbol{H}$ is negative semi-difinite. Such an $\boldsymbol{H}$ corresponds to some *dissimilarity data matrix*. The proof is trivial for DEDICOM, GIPSCAL, and other possible models for $\boldsymbol{H}$, since the decomposition defined by Eq. (1) is unique. Recently, Kiers and Takane (1992) have proved that a necessary and sufficient condition for these two models to have an equivalent expression is the positive-definiteness of $\boldsymbol{A}_s$ is Eq. (4). This is different from the condition for these two models to be expressible in terms of *f.d.c.* Hilbert space. Conclusion $C3$ is trivial, at this point, since $\varphi(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ is not necessarily positive (or negative) if $\boldsymbol{H}$ is indefinite.

We shall now examine the features of the Hermitian form used in DEDICOM, GIPSCAL, and HCM. It is apparent, from Eqs. (6), (7), and (13), that all of these models assume a symmetric bilinear form and a skew-symmetric bilinear form for the symmetric and the skew-symmetric parts of the proximity data, respectively. In the mathematical literatures these are also called the *metric tensor* and the *exterior product* or the *2-form.*

The metric tensor is a $\binom{0}{2}$ tensor at a point in a vector space, and thus a bilinear function of two vectors. Generally, a tensor of type $\binom{p}{q}$ at a point in a vector space (more generally at a point of a manifold) is defined to be a real-valued multilinear function which takes as arguments p *one-forms* and q vectors (Schutz, 1980). Here, a one-form is defined as a linear, real-valued function of vectors. Following Schutz, the metric tensor, G is defined by

$$G(\boldsymbol{\alpha}, \boldsymbol{\beta}) = G(\boldsymbol{\beta}, \boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}^t \boldsymbol{\beta}, \tag{15}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two vectors in an l-dimensional vector space. Moreover, let us define matrix $\boldsymbol{M}$ of order $(l \times l)$ whose (j, k) entry is

$$m_{jk} = G(\boldsymbol{b}_j, \boldsymbol{b}_k) \equiv \boldsymbol{b}_j^t \boldsymbol{b}_k, \tag{16}$$

where $\boldsymbol{b}_j$ $(j=1, 2, \cdots, l)$ constitute a basis of the vector space. The nonsingular symmetric matrix $\boldsymbol{M}$ can be diagonalized by using its eigenvalues and eigenvectors. If $\boldsymbol{M}$ is an identity matrix, the metric tensor is usually called the *Euclidean metric*, and the vector space is called the Euclidean space. More generally, the metric tensor is called a Euclidean metric if $\boldsymbol{M}$ is positive-definite or nagative-definite. Otherwise, it is called *indefinite*. A famous indefinite metric is a *Minkowski metric* in relativity theory in physics, where one eigenvalue is negative, and the others are all positive.

It is apparent that CASK has no metric tensor, since CASK can be written as

$$\boldsymbol{S}_{sk} = \boldsymbol{X\Gamma Y}^t = \boldsymbol{X\Gamma K X}^t, \tag{17}$$

where $\boldsymbol{\Gamma}$ is a diagonal matrix of order $(N \times N)$ of form

$$\boldsymbol{\Gamma} = \text{diag}\,(\gamma_1, \gamma_1, \gamma_2, \gamma_2, \cdots, (0)), \tag{18}$$

and $\boldsymbol{X}$ is an orthogonal matrix of order $(N \times N)$. Moreover, $\boldsymbol{K}$ is the matrix of order $(N \times N)$ which is defined by

$$\boldsymbol{K} = \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & (1) \end{bmatrix} \tag{19}$$

The second component of the Hermitian form used in DEDICOM, GIPSCAL, and HCM, is a certain 2-form. In general, an exterior form of degree k, or a k-form, is a function ω, of k vectors in an n-dimensional real vector space R^n, which has the following properties:

(1) *k-linear, i.e.,*

$$\omega(\lambda_1 \boldsymbol{\xi}_1' + \lambda_2 \boldsymbol{\xi}_1'', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k) = \lambda_1 \omega(\boldsymbol{\xi}_1', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k) + \lambda_2 \omega(\boldsymbol{\xi}_1'', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k), \tag{20}$$

(2) *antisymmetric, i.e.,*

$$\omega(\boldsymbol{\xi}_{i_1}, \boldsymbol{\xi}_{i_2}, \cdots, \boldsymbol{\xi}_{i_k}) = (-1)^{\nu} \omega(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k), \tag{21}$$

where

$$\nu = \begin{cases} 0, & \text{if the permutation } (i_1, i_2, \cdots, i_k) \text{ is even,} \\ 1, & \text{if the permutation } (i_1, i_2, \cdots, i_k) \text{ is odd.} \end{cases}$$

A well-known k-form is the oriented volume of the parallelepiped with edges $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\cdots$, $\boldsymbol{\xi}_k$ in an oriented Euclidean space E^k. Likewise, a famous 2-form is the oriented area of the parallelogram spanned by two vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ of the oriented Euclidean space E^2.

A 2-form can be written as a linear combination of the exterior products of two 1-forms. It should be noted that the value of the exterior product $\omega_1 \wedge \omega_2$ of two 1-forms ω_1 and ω_2 on the pair of vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in R^n$ can be thought of as a projection of the *oriented area* spanned by the two vectors in an n-dimensional real vector space onto the two-dimensional ω_1, ω_2-plane (Arnold, 1978). Figure 2.1 illustrates this idea. In this figure, $\omega(\boldsymbol{\xi}_1)$ and $\omega(\boldsymbol{\xi}_2)$ are the two-dimensional *images* of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ in the n-dimensional vector space.

It is apparent, from Eqs. (6), (7), (13), that $\boldsymbol{y}_j^t \boldsymbol{A}_{sk} \boldsymbol{y}_k$, $\boldsymbol{z}_j^t \boldsymbol{L}_q \boldsymbol{z}_k$, and $\boldsymbol{x}_j^t \boldsymbol{\Omega}_{sk} \boldsymbol{x}_k$ are all 2-forms and thus they can be a projection of the *oriented area* spanned by the two vectors in an n-dimensional real vector space onto a two-dimensional plane. Then, it will be natural to ask what are the differences of form among these three models. The answer is that *the variety of form depends upon what kind of framework we intend to choose to look at the projection of the oriented area, i.e.* the exterior product $\omega_1 \wedge \omega_2$. Mathematically, this is equivalent to saying what kind of basis we intend to choose. There are at least three strategies to do so. One is to choose an orthonormal basis on R^n and project the exterior

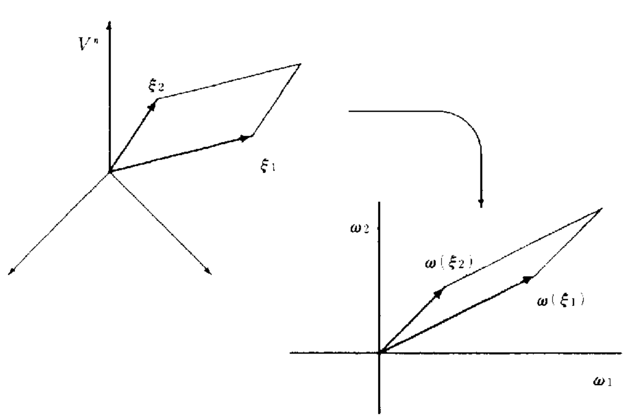

Fig. 2.1 $\omega_1 \wedge \omega_2$ as the image of the parallelogram

product onto $\binom{n}{2}$ component planes. GIPSCAL employs this strategy. Another is to take a symplectic basis on R^{2n}, and CASK and HCM use this strategy. A third is not to take a specific standard basis, and this is the strategy of DEDICOM.

The details of the first strategy can be explained as follows: In this case, we first consider q independent 1-forms $x_1, x_2, \cdots, x_q$, which are called the *basic forms.* The 2-forms $x_i \wedge x_j$ constructed from these basic forms have a very simple geometrical meaning (Arnold, 1978): the value of $x_i \wedge x_j$ is the oriented area of the image of the parallelogram spanned by the two vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ on the coordinate plane x_i, x_j under the projection parallel to the remaining coordinate functions.

It is interesting to note that (a) $\binom{q}{2}$ $(=q(q-1)/2)$ 2-forms, $x_i \wedge x_j$, are linearly independent, and (b) every 2-form on the q-dimensional space with coordinate $x_1, \cdots, x_q$ can be uniquely represented in the form

$$\omega^2 = \sum_{i<j}^{q} a_{ij} \, (x_i \wedge x_j), \tag{22}$$

where

$$a_{ij} = \omega_b^2(\boldsymbol{e}_i, \boldsymbol{e}_j), \tag{23}$$

and $\boldsymbol{e}_i$ is the i-th basis vector (Arnold, 1978). The ω_b^2 is the 2-form on the pair $\boldsymbol{e}_i$, $\boldsymbol{e}_j$. It is evident that the skew-symmetric part of GIPSCAL, *i.e.* $\boldsymbol{z}_j^t \boldsymbol{L}_q \boldsymbol{z}_k$, is nothing but the 2-form ω^2 in Eq. (22) in the special case when the q basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_q$ are orthonormal.

The second strategy is to employ a symplectic basis. According to Arnold (1978):

A symplectic (linear) structure on R^{2n} is defined as a nondegenerate bilinear skew-symmetric 2-form given in R^{2n}. This form is called the *skew-scalar product* and is denoted by $[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2] = -[\boldsymbol{\eta}_2, \boldsymbol{\eta}_1]$. The space R^{2n}, together with the symplectic structure $[\ ,\ ]$, is called a *symplectic vector space.* Here, a 2-form $[\ ,\ ]$ on R^{2n} is called nondegenerate if, for any $\boldsymbol{\eta}_2$, $[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]=0$ then $\boldsymbol{\eta}_1 = \boldsymbol{o}$.

In general a skew-scalar product, $[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2] = \omega^2(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ may take a variety of forms depending on a basis in a symplectic vector space. However, if we take a suitable basis, it is represented by a very simple form

$$\omega^2 = x_1 \wedge y_1 + \cdots + x_n \wedge y_n, \tag{24}$$

where $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be coordinate functions on R^{2n} and this symplectic structure is called the *standard symplectic structure.* It is apparent, from the form of ω^2 is Eq. (24), that in this case $2n$ basis vectors $\boldsymbol{e}_{x_i}$ and $\boldsymbol{e}_{y_i}$ $(i=1, \cdots, n)$ satisfy the relations

$$[\boldsymbol{e}_{x_i}, \boldsymbol{e}_{x_j}] = [\boldsymbol{e}_{y_i}, \boldsymbol{e}_{y_j}] = [\boldsymbol{e}_{x_i}, \boldsymbol{e}_{y_j}] = 0, \quad (i \neq j), \tag{25}$$

and

$$[\boldsymbol{e}_{x_i}, \boldsymbol{e}_{y_i}] = 1. \tag{26}$$

The basis vectors which satisfy Eqs. (25) and (26) constitute a *standard symplectic basis.*

It is apparent from the definition that in the standard symplectic structure the oriented area ω^2 spanned by the two vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ is projected on the n coordinate planes (x_i, y_i), $i=1, \cdots, n$. As a result, the ω^2 is equal to the sum of the oriented areas of the parallelogram on the n coordinate planes.

It is apparent from the above discussion that CASK employs the standard symplectic basis and that matrix $\boldsymbol{K}$ in Eq. (19) is nothing but the expression of Eqs. (25) and (26) if the order of $\boldsymbol{\Gamma}$ defined by Eq. (18) is even.

The geometry of a symplectic space is different from that of a Euclidean space and it is evident that the symplectic structure itself has no Euclidean metric structure. However, we can, of course, impose some Euclidean structure on the symplectic space as follows (Arnold, 1978):

$$\boldsymbol{x}^t\boldsymbol{x}=\sum_{i=1}^{n}(x_i^2+y_i^2), \tag{27}$$

where

$$\boldsymbol{x}=\sum_{i=1}^{n}(x_i\boldsymbol{e}_{x_i}+y_i\boldsymbol{e}_{y_i}). \tag{28}$$

This is equivalent to introduce a complex space C^n and define the Hermitian scalar product

$$\boldsymbol{\eta}_1^t\boldsymbol{\eta}_2+i[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2], \tag{29}$$

in the space (Arnold, 1978). It is easy to see that the Hermitian scalar product defined by Eq. (13) is equivalent to that defined here by Eq. (29), if $\boldsymbol{H}$ is positive (negative) semi-definite. These examinations lead to conclusion $C4$.

Conclusion $C4$ suggests that DEDICOM, GIPSCAL, and HCM can take alternative forms depending upon the basis chosen. In fact, we can also choose, for example, a symplectic basis for DEDICOM and GIPSCAL. Let us first rewrite Eq. (4), defining an orthogonal transformation $\boldsymbol{O}$ such that

$$\boldsymbol{O}\boldsymbol{A}_s\boldsymbol{O}^t=\boldsymbol{D}_s, \tag{30}$$

and

$$\boldsymbol{Y}=\boldsymbol{P}\boldsymbol{O}, \tag{31}$$

as

$$\boldsymbol{H}=\boldsymbol{P}\boldsymbol{D}_s\boldsymbol{P}^t+i\ \boldsymbol{P}\boldsymbol{O}\boldsymbol{A}_{sk}\boldsymbol{O}^t\boldsymbol{P}^t. \tag{32}$$

If $\boldsymbol{D}_s$ is positive definite, Eq. (32) becomes

$$\boldsymbol{H}=(\boldsymbol{P}\boldsymbol{D}_s^{\frac{1}{2}})\boldsymbol{P}\boldsymbol{D}_s^{\frac{1}{2}})^t+i(\boldsymbol{P}\boldsymbol{D}_s^{\frac{1}{2}})\boldsymbol{D}_s^{-\frac{1}{2}}\boldsymbol{O}\boldsymbol{A}_{sk}\boldsymbol{O}^t\boldsymbol{D}_s^{-\frac{1}{2}})(\boldsymbol{P}\boldsymbol{D}_s^{\frac{1}{2}})^t. \tag{33}$$

Then, as Kiers and Takane (1992) have done, we can decompose the "core" matrix via a singular value decomposition into

$$\boldsymbol{D}_s^{\frac{1}{2}}\boldsymbol{O}\boldsymbol{A}_{sk}\boldsymbol{O}^t\boldsymbol{D}_s^{-\frac{1}{2}}=\boldsymbol{T}\boldsymbol{\Sigma}\boldsymbol{K}\boldsymbol{T}^t=\boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^t, \tag{34}$$

where $\boldsymbol{T}$ is an orthogonal matrix of order $(p\times p)$. Then, Eq. (33) becomes

$$\boldsymbol{H}=\boldsymbol{Q}\boldsymbol{Q}^t+i\boldsymbol{Q}\boldsymbol{J}\boldsymbol{Q}^t, \tag{35}$$

where

$$\boldsymbol{Q}=\boldsymbol{P}\boldsymbol{D}_{s^{\frac{1}{2}}}\boldsymbol{T}. \tag{36}$$

It should be noted that if $\boldsymbol{H}$ is negative semi-definite we may merely change the sign of $\boldsymbol{H}$. To prove conclusion $C5$, we must further administer a coordinate transformation to $\boldsymbol{Q}$ in Eq. (35). Letting

$$\boldsymbol{R}=\boldsymbol{Q}\boldsymbol{\Sigma}^{\frac{1}{2}}, \tag{37}$$

and noticing the relation in Eq. (34),

$$\boldsymbol{\Sigma K}=\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{K}\boldsymbol{\Sigma}^{\frac{1}{2}}=\boldsymbol{J}, \tag{38}$$

we get

$$\boldsymbol{H}=\boldsymbol{R}\boldsymbol{\Sigma}^{-1}\boldsymbol{R}^{t}+i\ \boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{t}. \tag{39}$$

This proves conclusion $C5$ for DEDICOM. The same is true for GIPSCAL.

3. An interesting property of f.d.c. Hilbert space models

Unlike the distance model, the Hermitian scalar product model has an interesting property in that the similarity between the pair of objects located far from the centroid of objects, say, the origin, is greater than that located near the origin, even if their distances are the same. In other words, if this model holds, then subjects are likely to underestimate the distance between the pairs of objects located far from the origin and overestimate the distance between the pairs located near the origin.

To prove it, we shall first point out that for a complex pre-Hilbert space (thus, also for the f.d.c. Hilbert space), the *polar identity*

$$\begin{aligned}\varphi(\boldsymbol{\zeta},\ \boldsymbol{\tau})&=\frac{1}{4}(\|\ \boldsymbol{\zeta}+\boldsymbol{\tau}\ \|^2-\|\ \boldsymbol{\zeta}-\boldsymbol{\tau}\ \|^2)+\frac{1}{4}\,i(\|\ \boldsymbol{\zeta}+i\boldsymbol{\tau}\ \|^2-\|\ \boldsymbol{\zeta}-i\boldsymbol{\tau}\ \|^2),\\&=\frac{1}{2}(\|\ \boldsymbol{\zeta}\ \|^2+\|\ \boldsymbol{\tau}\ \|^2-\|\ \boldsymbol{\zeta}-\boldsymbol{\tau}\ \|^2)+\frac{1}{2}\,i(\|\ \boldsymbol{\zeta}\ \|^2+\|\ \boldsymbol{\tau}\ \|^2-\|\ \boldsymbol{\zeta}-i\boldsymbol{\tau}\ \|^2),\end{aligned} \tag{40}$$

holds (Cristescu, 1977), where $\varphi(\boldsymbol{\zeta},\ \boldsymbol{\tau})$ is a Hermitian form. Then, remembering that $h_{jk}=\varphi(\boldsymbol{v}_j,\ \boldsymbol{v}_k)$, which we have proved in the second section in leading to conclusion $C1$, we get

$$\begin{aligned}h_{jk}=&\frac{1}{2}(\|\ \boldsymbol{v}_j\ \|^2+\|\ \boldsymbol{v}_k\|^2-\|\ \boldsymbol{v}_j-\boldsymbol{v}_k\ \|^2)\\&+\frac{1}{2}\,i(\|\ \boldsymbol{v}_j\ \|^2+\|\boldsymbol{v}_k\ \|^2-\|\ \boldsymbol{v}_j-i\boldsymbol{v}_k\ \|^2).\end{aligned} \tag{41}$$

Then, in terms of Eq. (1), we get

$$\begin{aligned}s_{jk}=&\frac{1}{2}(\|\ \boldsymbol{v}_j\ \|^2+\|\ \boldsymbol{v}_k\ \|^2-\|\ \boldsymbol{v}_j-\boldsymbol{v}_k\ \|^2)\\&+\ \frac{1}{2}(\|\ \boldsymbol{v}_j\ \|^2+\|\ \boldsymbol{v}_k\ \|^2-\|\boldsymbol{v}_j-i\boldsymbol{v}_k\ \|^2),\end{aligned} \tag{42}$$

Rearranging the right-hand side of Eq. (42), we have

$$s_{jk}=-\frac{1}{2}\{\| \boldsymbol{v}_j-\boldsymbol{v}_k \|^2+\| \boldsymbol{v}_j-i\boldsymbol{v}_k \|^2\}+(\| \boldsymbol{v}_j \|^2+\| \boldsymbol{v}_k \|^2). \tag{43}$$

Notice here that in general $\| \boldsymbol{v}_j-i\boldsymbol{v}_k \|^2$ is not equal to $\| \boldsymbol{v}_k-i\boldsymbol{v}_j \|^2$.

Equation (43) represents the property under study, if we assume that the centroid of coordinates of objects is at or near the origin. For in this case the second term on the right-hand side of Eq. (43) increases as the objects j and k are away from the origin.

For a real Hilbert space (*i.e.*, a Euclidean space), it is well-known that,

$$s_{jk}=-\frac{1}{2}\| \boldsymbol{v}_j-\boldsymbol{v}_k \|^2+\frac{1}{2}(\| \boldsymbol{v}_j \|^2+\|\boldsymbol{v}_k \|^2). \tag{44}$$

It is apparent, from Eq. (44), that the same discussion as in the above asymmetric matrix holds for the symmetric proximity matrix.

4. Generalization of the Young-Householder theorem

Conclusion $C2$ in the second section motivates the following theorem. This theorem is an extension of the Young-Householder theorem into f.d.c. Hilbert space.

THEOREM *Let* $\boldsymbol{v}_j (1\leq j\leq N)$ *be the row vector of order n, which corresponds to the j-th row of* $\boldsymbol{U}_1$ *in Eq. (9). Put*

$$d_{jk}=\| \boldsymbol{v}_j-\boldsymbol{v}_k \|, \quad 1\leq j,k\leq N, \tag{45}$$

$$d_{jo}=\| \boldsymbol{v}_j \|, \quad f\leq j\leq N, \tag{46}$$

and

$$\bar{d}_{jk}=\| \boldsymbol{v}_j-i\boldsymbol{v}_k \|, \quad 1\leq j,k\leq N. \tag{47}$$

Then the following equalities hold ;

$$d_{jk}=d_{kj}, \quad 1\leq j,k\leq N, \qquad d_{jj}=0, \quad 1\leq j\leq N, \tag{48}$$

$$d_{jo}^2+d_{ko}^2-\bar{d}_{jk}^2=-(d_{ko}^2+d_{jo}^2-\bar{d}_{kj}^2), \quad 1\leq j,k\leq N, \tag{49}$$

$$h_{jk}=\frac{1}{2}(d_{jo}^2+d_{ko}^2-d_{jk}^2)+\frac{1}{2}i(d_{jo}^2+d_{ko}^2-\bar{d}_{jk}^2), \quad 1\leq j,k\leq N. \tag{50}$$

Here h_{jk} *is given by Eqs. (6), (7), or (13) and the matrix* $\boldsymbol{H}=\{h_{jk}\}$, $(1\leq j,\ k\leq N)$ *is a positive semi-definite Hermitian matrix.*

Conversely, if we are given positive real numbers d_{jk} $(1\leq j\neq k\leq N)$ *satisfying Eqs. (48) and (49) for a suitable choice of non-negative real numbers* $\bar{d}_{jk}$ *and* $d_{jo}(1\leq j,\ k\leq N)$, *then the set* $\{d_{jk}\}$ *gives the mutual distances of a real (true) set of points in an f.d.c. Hilbert space if and only if* $\boldsymbol{H}$ *is positive semi-definite. The set of points is unique apart from an arbitrary unitary transformation.*

PROOF We can prove the theorem in a similar manner to that given by Young and Householder (1938). The only difference is that in this case we must use Eq. (50), which is another expression of Eq. (41), instead of Eq. (1) of Young and Householder (1938). Equation (49) is merely a condition for $h_{kj}=\bar{h}_{jk}$, $\bar{h}_{jk}$ being the complex conjugate of h_{jk}.

The proof of the latter part of the theorem can be stated in another way. Let us first note that Eqs. (48), (49), and (50) hold for the mutual distance of a true set of points in an f.d.c. Hilbert space. Second, if $\boldsymbol{H}$ is positive semi-difinite, we can define the following distances using the row vectors $\boldsymbol{v}'_j$, $\boldsymbol{v}'_k$ corresponding to the j-th row and the k-th row of $\boldsymbol{U}_1$ in Eq. (9):

$$d'_{jk}=\|\boldsymbol{v}'_j-\boldsymbol{v}'_k\|,\quad 1\le j,\, k\le N, \tag{51}$$

$$d'_{jo}=\|\boldsymbol{v}'_j\|,\quad 1\le j\le N, \tag{52}$$

and

$$\bar{d}'_{jk}=\|\boldsymbol{v}'_j-i\boldsymbol{v}'_k\|,\quad 1\le j,\, k\le N. \tag{53}$$

Then, from the result of the first part of the theorem we have

$$d'_{jk}=d'_{kj},\quad 1\le j,\, k\le N,\qquad d'_{jj}=0,\quad 1\le j\le N, \tag{54}$$

$$d'^2_{jo}+d'^2_{ko}-\bar{d}'^2_{jk}=-(d'^2_{ko}+d'^2_{jo}-\bar{d}'^2_{kj}),\quad 1\le j,\, k\le N, \tag{55}$$

$$h_{jk}=\frac{1}{2}(d'^2_{jo}+d'^2_{ko}-d'^2_{jk})+\frac{1}{2}\,i(d'^2_{jo}+d'^2_{ko}-\bar{d}'^2_{jk}),\quad 1\le j,\, k\le N. \tag{56}$$

From Eqs. (49) and (50), we have

$$h_{jj}=d^2_{jo},\quad 1\le j\le N. \tag{57}$$

Moreover, from Eqs. (55) and (56) we get

$$h_{jj}=d'^2_{jo},\quad 1\le j\le N. \tag{58}$$

Then we have $d^2_{jo}=d'^2_{jo}$, which means that

$$d_{jo}=d'_{jo},\quad 1\le j\le N. \tag{59}$$

Next, comparing the real parts and the imaginary parts of Eqs. (50) and (56), we get

$$d^2_{jo}+d^2_{ko}-d^2_{jk}=d'^2_{jo}+d'^2_{ko}-d'^2_{jk},\quad 1\le j,\, k\le N, \tag{60}$$

$$d^2_{jo}+d^2_{ko}-\bar{d}^2_{jk}=d'^2_{jo}+d'^2_{ko}-\bar{d}'^2_{jk},\quad 1\le j,\, k\le N. \tag{61}$$

Equations (59) and (60) yield $d^2_{jk}=d'^2_{jk}$, from which we get

$$d_{jk}=d'_{jk}. \tag{62}$$

Likewise, from Eqs. (59) and (61) we get

$$\bar{d}_{jk}=\bar{d}'_{jk} \tag{63}$$

The invariance of the coordinates of points over a unitary transformation is evident. ■

5. Discussion

Throughout this paper we have exclusively dealt with the error-free data, *i.e.* the error-free proximity matrices $\boldsymbol{S}$ as well as the corresponding Hermitian matrices $\boldsymbol{H}$ measured at a ratio level. However, in a practical situation, both of these matrices may be fallible and not necessarily be measured at the ratio level. Furthermore, we can neither observe nor estimate the special distances defined by Eq. (47), since it is not concerned with the direct distance between two points in an f.d.c. Hilbert space in marked contrast to the distance in classical MDS. In such a case we must estimate them from the data. If the proximity judgments are measured at the ratio level, there are no missing observations, and the matrix $\boldsymbol{H}$ is positive semi-definite, we can use HCM as one such method. For in such a case HCM is also solved by a singular value decomposition and thus has a least squares (LS) property according to the Schmidt-Mirsky theorem, which is a generalization of the famous Eckart-Young theorem (Stewart & Sun, 1990, p. 205). Otherwise, we must estimate them using some scaling procedure. GIPSCAL and its generalized version proposed recently by Kiers and Takane (1992) are thought of as two such procedures, which indirectly estimate these h_{jk}'s by LS criteria. Recently, Chino (1992) has proposed an ML procedure for estimating and testing parameters of a general Hermitian form model, including all these discussed in this paper, given proximity or dissimilarity data measured at least at an ordinal level. The Hermitian form model discussed in this paper may also be applicable to square contingency tables, as Chino (1991b) has pointed out.

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